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On the hydrodynamics of pairs of spheres falling along their line of centres in a viscous medium

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The velocities, accelerations and drag forces experienced by two equal spheres falling along their line of centres in a viscous fluid were determined for three groups of Reynolds numbers R in the range where it is commonly assumed that Stokes's approximation applies. For all groups, with R ranging between 0·060 and 0·216, both spheres continually acclerated as they fell, and the upper sphere fell faster and accelerated more than the lower one. In contrast to Stimson & Jeffery's (1926) theory, which is based on the Stokes approximation, and to most earlier experimenters, the drag-force coefficients of the upper sphere computed from the experiments were significantly smaller than those for the lower sphere. Oseen's theory for this case agreed with the experiments in some respects, but contrary to it the drag-force coefficient varied with R for the upper sphere as well as the lower sphere.

1. Introduction

The application of hydrodynamic theory to the behaviour of solid and liquid particles moving in a viscous medium at low Reynolds numbers has received increased attention in recent years in connexion with problems in chemical engineering, geological engineering, air chemistry, and cloud and precipitation physics. The studies of colloids and aerosols, for instance, require solutions of the many-body problem.

In cloud physics the problem enters importantly in considering drop growth by accretion. The drops are sufficiently sparse so that ordinarily attention can be confined to the interaction of two of them at a time, and the rate at which precipitation-sized drops can form by collection of cloud drops depends critically on the collision efficiency, which measures the probability that a drop below the collector drop will collide with it rather than being carried around it by the viscous drag of the medium. In the case of two drops with centres vertically one above the other collision would occur unless they move with equal velocities, but for sufficiently large Reynolds numbers the asymmetry of the flow assures unequal velocities even for equal spheres.

Studies have shown that on the one hand clouds formed by condensation—for instance, in the updraft in a convective cloud—tend to consist of small drops, modal radius between 5 and 10 μm, with relatively little dispersion of sizes, i.e.

most of the drops are nearly equal; and on the other hand the collision efficiency for drops smaller than 20 μm is practically zero (Hocking 1959; Davis & Sartor, 1967). The conclusion derived from these studies is that clouds require larger drops or ice crystals to produce precipitation.

The indication that the collision efficiency for equal drops is very large (Telford, Thorndyke & Bowen 1955; Woods & Mason 1965) led us to look into the question, for how small drops will the asymmetry of flow lead to non-zero collision efficiencies for equal drops? To study this question we have carried out experiments with pairs of equal solid spheres falling in oil.

The theoretical and experimental studies previously reported on the interaction of spheres falling in a viscous medium have been summarized by Happel & Brenner (1965). It is apparent from their discussion that the theoretical treatment of this subject involves great mathematical difficulties, even for the case of only two spheres in an unbounded incompressible fluid. Consequently the problem has been solved only when certain simplifying and restrictive assumptions are made, and then usually only approximately. Among the investigators who have dealt with the problem of two spheres moving in a viscous fluid are Smoluchowski (1911, 1912), Faxén & Dahl (1925), Stimson & Jeffery (1926), Oseen (1927), Burgers (1941, 1942, 1943), Hocking (1959) and Kynch (1959).

For the case of spheres moving without acceleration along the line of centres, Stimson & Jeffery solved the problem rigorously on the assumption of vanishingly small Reynolds number R, so that Stokes's approximation holds, i.e. the non-linear terms in the equation of motion of the fluid may be neglected. For equal spheres their result gives equal drag on both spheres. Their solution gives for the ratio of the drag to that of an isolated sphere moving with the same velocity:

$$\Lambda_{SJ} = \tfrac{4}{3}\sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)}\left(1-\frac{4\sinh^2(n+\frac{1}{2})\alpha-(2n+1)^2\sinh^2\alpha}{2\sinh(2n+1)\alpha+(2n+1)\sinh 2\alpha}\right),$$

where $\cosh\alpha = l/2A$.

Oseen solved the problem approximately using the approximation to the non-linear terms which bears his name. His results give different values for the drag on the rear and forward spheres. For equal spheres moving along their line of centres they are respectively

$$\Lambda_0 = 1-\frac{3}{2}\frac{A}{l}, \quad \lambda_0 = 1-\frac{3}{2}\frac{A}{l}\frac{(1-e^{-\beta})}{\beta} \cong 1-\frac{3}{2}\frac{A}{l}+\frac{3}{8}R,$$

where $\beta = (R/2)(l/A)$ and the other terms are defined in §3. For this case Oseen's drag on the rear sphere is independent of the Reynolds number, while the drag on the forward sphere is increased by an amount which is approximately proportional to R. Curves showing λ_0 for three values of R and Λ_{SJ} are shown in figure 2.

Experimental studies of this problem also have been unsatisfactory because of the numerous difficulties involved in conducting experiments which give sufficiently accurate results. Most experimenters attempted to measure the velocities of two spheres falling in a viscous fluid at low Reynolds numbers, and from them to evaluate the drag on each. They include Hall (1956), Eveson, Hall & Ward

(1959), Pfeffer (1958), Bart (1959), Happel & Pfeffer (1960) and Isaakyan & Gasparyan (1966). Their results are represented in figure 1. Except for Isaakyan & Gasparyan they found that the velocities of both spheres, while larger than that of an isolated sphere, were equal. Telford & Cottis (1964) attempted to measure the drag forces on pairs of spheres directly. They found differences between the forces on the forward and rear spheres, but these differences, reproduced in figure 5, do not vary reasonably with the distance between the spheres. The marked differences among the previous experimental results, particularly between Isaakyan & Gasparyan's and the others, led us to undertake experiments of our own.

In this paper we shall present the results of some experiments which are more accurate than those reported previously, and compare these results with available theories in order to see the extent to which the simplifying assumptions are justified. We shall restrict ourselves to the case of two equal spheres falling along the line of centres in the range of Reynolds numbers for which the Stokes approximation is normally regarded as holding. It will be shown that there are marked deviations from the theoretical expectations.

2. Experimental procedures

Each run in the experiment consisted of dropping two equal solid spheres in succession along a vertical line in a 12 in. diameter Plexiglas tank containing a transparent oil and detecting the times they passed through a series of horizontal light beams. Approximately 50 runs were carried out to cover various Reynolds numbers and a range of distances between spheres from about 12 radii practically to contact between them.

The critical factors in the experiment are (*i*) dropping the spheres exactly in the vertical, and (*ii*) the maintenance of constant viscosity which is accurately determined.

The first of these was achieved by using a specially built precision mechanism in which each sphere was held by two arms ending in knife-edge pins in a position controlled by micrometer screws. The arms could be raised and lowered without affecting the horizontal positions of the spheres. The two spheres were placed in the arms and positioned so that their centres were as nearly as possible in a vertical line, and then the arms were lowered into the oil. A time-delay relay provided the possibility of releasing the upper sphere at a variable time interval after the lower. This enabled the initial separation of the spheres to be varied.

Several methods were used to estimate the precision with which the spheres could be positioned in a vertical line. While a strict evaluation of the deviation proved impossible, it is estimated that the deviation was always less than 0·3 mm, which was about one-twentieth the diameter of the smallest sphere used.

The oil used was Altavis 210, which has density 0·8685 g/cm^3 and kinematic viscosity 53·90 Stokes at 20 °C. While this oil was chosen because its viscosity varies less with temperature than most other fluids, a change of 0·05 °C corresponds to almost 0·3 % in viscosity in the temperature range occurring in the experiment. For this reason the temperature was not allowed to vary more than this amount during a run. Before each experiment the oil was circulated through an external pumping system to stir it thoroughly. During each run the temperature was monitored with thermocouples and if it changed more than 0·05 °C the run was terminated.

The position of the spheres as a function of time was determined by a series of ten horizontal light beams approximately 5 cm apart. The vertical distances between the light beams at the centre of the tank were determined to within 0·01 cm by several means, including use of a single sphere of extremely small Reynolds number falling at terminal velocity and the lowering of objects attached to wires the length of which could be measured accurately. The light beams were positioned in a vertical plane using a plumb on a fine wire. The interruptions of each light beam as the spheres fell were detected by a photocell at the side of the tank opposite the source. The interruption actuated a signal which was recorded on a dual-channel strip chart recorder, the second channel of which recorded a 100 Hz signal. In this way the time each sphere passed each light beam was determined to an accuracy of 5×10^{-3} sec.

Experiments were carried out using spheres of the following materials and diameters—stainless steel, $\frac{1}{4}$ in. (0·635 cm) and $\frac{5}{16}$ in. (0·795 cm); tungsten carbide, $\frac{7}{32}$ in. (0·555 cm), $\frac{1}{4}$ in. (0·635 cm) and $\frac{9}{32}$ in. (0·714 cm). The spheres were carefully checked to make sure that the pair numbers in each run were identical. For instance, the masses of the members of a pair differed by less than 0·05 %. The diameters were accurate to within 10^{-4} in.

No correction for wall effect was applied in our data. We recognized that for the sizes of spheres and tank used the theoretical expressions for wall correction for the velocity of a single sphere is about 4 %, and that it has been suggested (Happel & Pfeffer 1960) that the correction for a pair of spheres is about twice that for a single sphere. However, these theories were derived using the Stokes approximation, and since our experiments very early showed it not to be valid even for the smallest Reynolds numbers used, we considered it inappropriate to apply the correction to our data. Fidleris & Whitmore (1961) showed that for a single sphere the wall correction goes down with increasing Reynolds number. Our velocity measurements thus should be corrected by some factor smaller than 8 %. How much smaller we cannot say, but it is likely that it is not much larger than the 1 % innate uncertainty of our measurements.

3. Evaluation of the data

Each run gave the time of passage of the spheres at fixed positions along the vertical. To compute the velocities and accelerations, and in particular to compare the viscous-drag forces on the two spheres as a function of the distance between them, it was necessary to interpolate between successive positions and times. Plots of the distance versus time suggested that the data could be represented by polynomials. To allow for variations of accleration with time, fourth-order polynomials were used. The coefficients were evaluated for each run by the least squares method, using the IBM 7094 computer.

From the polynomial expressions for the positions it was possible to determine the velocities and accelerations of the spheres at every moment and the centre-to-centre separation. To evaluate the drag forces it is necessary to consider the equations of motion of the spheres.

We shall use the following notations:

A = radius of sphere,
ρ_s = density of spheres,
ρ_m = density of oil,
μ, ν = dynamic and kinematic viscosity of oil,
$V, \dot{V}$ = velocity and acceleration of upper sphere,
$v, \dot{v}$ = velocity and acceleration of lower sphere,
$R \equiv 2Av/\nu$, the Reynolds number (of the lower sphere),
F, f = drag force on upper and lower sphere,
$m = (4/3)\pi A^3 \rho_s$ = mass of spheres,
$m'g \equiv (4/3)\pi A^3(\rho_s - \rho_m)g$, net gravitational force on spheres,
$F_s, f_s = 6\pi\mu A V$, Stokes law drag force on upper and lower sphere,
$U = 2A^2(\rho_s - \rho_m)g/9\mu$ terminal velocity of single sphere falling under Stokes drag force,
$F_\infty = 6\pi\mu A U = m'g$, Stokes drag force on isolated sphere falling at terminal velocity,
$\Lambda \equiv F/F_s$, $\lambda \equiv f/f_s$,
l = distance between centres of spheres.

The equations of motion of the spheres are then

$$m\dot{V} = m'g - F, \quad m\dot{v} = m'g - f.$$

If we have evaluated $\dot{V}$ and $\dot{v}$ experimentally we can evaluate F and f from these equations.

Since the interaction between the spheres was expected to be a function of the distance between them, the various results were plotted against the separation l. Distances were expressed non-dimensionally by dividing by A, velocities by dividing by U, and accelerations by dividing by $m'g/m$.

4. Results

For all sizes of spheres used in the experiment, with R ranging down to 0·06, the upper sphere moved faster than the lower one and tended to overtake it, and both of them accelerated as the distance between them decreased. Because of the changing velocities the value of R is different for each sphere and changes as the run proceeds. Consequently the results of each run cannot be identified with a single R. For this reason the results were grouped by ranges of R, using the values for the lower sphere. The groups used are (*i*) $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$, (*ii*) $0{\cdot}085 \leqslant R \leqslant 0{\cdot}150$, and (*iii*) $0{\cdot}169 \leqslant R \leqslant 0{\cdot}216$. To obtain the average curve for each group the curves for velocity, acceleration and drag force for the individual runs were plotted against l/A, and average values computed for specified values of l/A.

It should be mentioned that V/U and v/U depended only on R and l/A and were independent of whether steel or tungsten carbide spheres were used.

The results are shown in figures 1 to 5. In these figures are represented the velocities, accelerations, and drag forces and their differences (lower to upper sphere) as functions of distance between the spheres.

In figure 1 the velocities of the spheres for the three groups are shown together with the results of previous experiments. The general pattern of our results conforms to those of the earlier workers. The velocities of both spheres falling in a

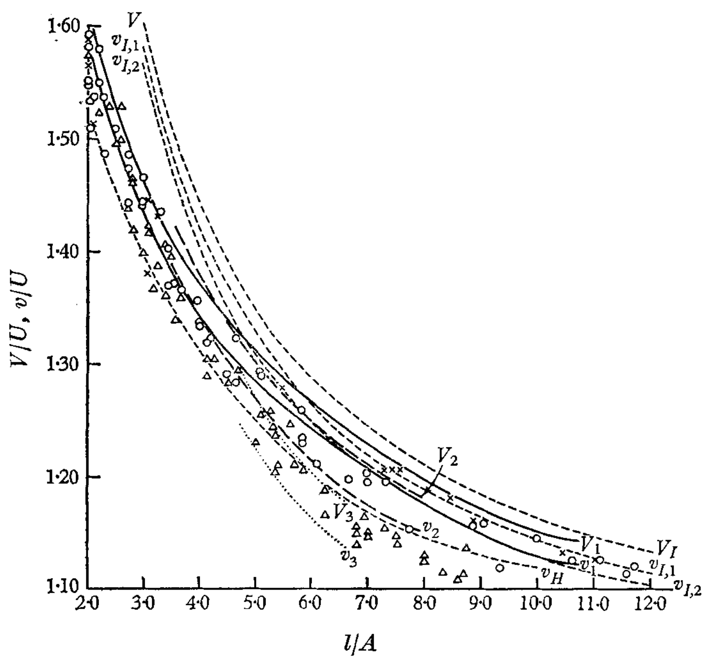

FIGURE 1. Variation of the velocity with distance between spheres. Results of present experiments and experiments cited in literature.

– – –, results in literature. V_I, Isaakyan & Gasparyan, upper. $v_{I,1}$, Isaakyan & Gasparyan, lower, $R = 0{\cdot}08$. $v_{I,2}$, Isaakyan & Gasparyan, lower, $R = 0{\cdot}16$. v_H, Hall and Eveson *et al.*, $R \leqslant 0{\cdot}10$. ×, Bart, $0{\cdot}015 \leqslant R \leqslant 0{\cdot}050$. ○, Pfeffer and Happel & Pfeffer, $0{\cdot}008 \leqslant R \leqslant 0{\cdot}028$. Δ, Pfeffer and Happel & Pfeffer, $0{\cdot}049 \leqslant R \leqslant 0{\cdot}187$.

——, present results. V_1, upper sphere; v_1, lower sphere; $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$. – –, V_2, upper sphere; v_2, lower sphere; $0{\cdot}085 \leqslant R \leqslant 0{\cdot}150$. . . . , V_3, upper sphere; v_3, lower sphere; $0{\cdot}169 \leqslant R \leqslant 0{\cdot}216$.

pair are greater than the terminal velocity of an isolated sphere, and the excess is greater the smaller their separation. Like Isaakyan & Gasparyan's results, but unlike the others, the upper sphere fell faster than the lower in all groups, and the lower sphere fell faster for smaller R than for larger R for l/A greater than five. Unlike Isaakyan & Gasparyan's results, the upper sphere also fell faster for smaller R than for larger R for l/A greater than five, and both V and v tend to be smaller for smaller R than for larger R for smaller values of l/A than four.

Corresponding to the increase of velocities with decrease in l/A the drag-force coefficients (figure 2) decrease as l/A decreases. The excess speed of the upper sphere (figure 3) increases as the spheres get closer together, as do the accelerations (figure 4) and the difference in drag (figure 5).

The variation with R also changes as the spheres get closer together. At values of l/A greater than four or five the drag-force coefficients Λ and λ are reduced most by the presence of the second sphere for the lowest Reynolds number group; at smaller l/A the curves for the larger Reynolds number groups have a larger slope and tend to cross toward lower values. The accelerations are larger for larger R; the values for group (*iii*) were too high to fit in figure 4. For small

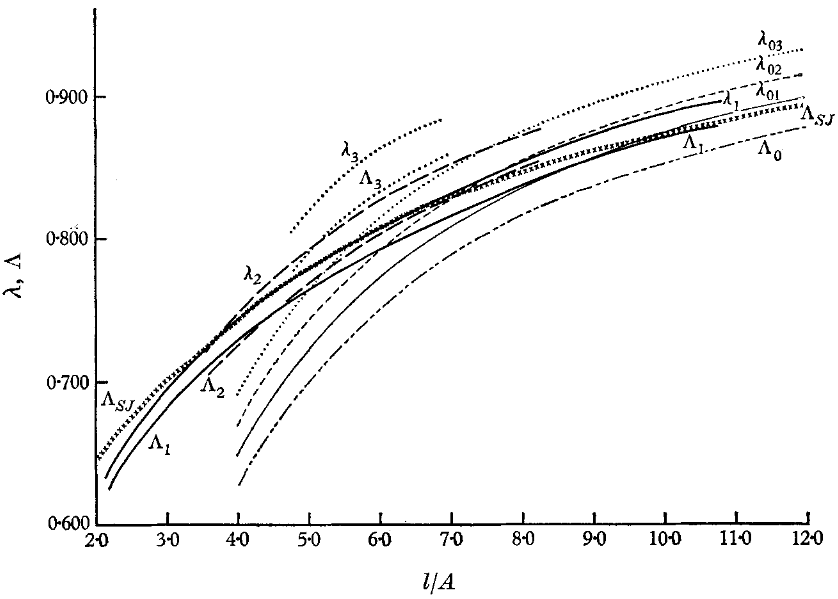

FIGURE 2. Variation of the drag coefficient with distance between spheres of a pair. Comparison of present experimental results with theoretical results of Oseen and Stimson & Jeffery.

Theory, Λ_{SJ}: Stimson & Jeffery. Λ_0, Oseen, upper sphere. λ_{01}, Oseen, lower sphere; $R = 0{\cdot}06$. λ_{02}, Oseen, lower sphere; $R = 0{\cdot}12$. λ_{03}, Oseen, lower sphere; $R = 0{\cdot}20$.

Experiment: present results. Λ_1, upper sphere; λ_1, lower sphere; $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$. Λ_2, upper sphere; λ_2, lower sphere; $0{\cdot}085 \leqslant R \leqslant 0{\cdot}150$. Λ_3, upper sphere; λ_3, lower sphere; $0{\cdot}169 \leqslant R \leqslant 0{\cdot}216$.

values of l/A the acceleration is much larger in group (*ii*) than in group (*i*), but as l/A increases the acceleration in group (*ii*) decreases more rapidly than in group (*i*), so that both approach zero. The difference between upper and lower sphere velocities also increases with R; again this effect is more pronounced at small l/A values. In general, the effect of changing R is much larger, for all the quantities studied, when the spheres are close together.

The experiments demonstrate clearly that the motion of the upper sphere is different from that of the lower one, for the l/A range investigated. It was found that the drag force on the upper sphere is always less than on the lower sphere; consequently the velocity and acceleration of the upper sphere are always larger than the corresponding values of the lower sphere. These differences increase for small separations for all the three groups and for a fixed separation they increase with R.

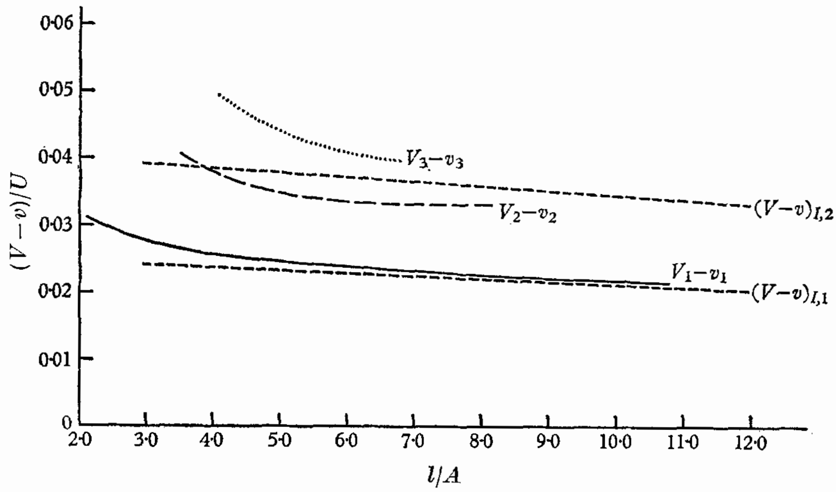

FIGURE 3. Difference between velocities of upper and lower sphere of a pair as function of distance between them. Comparison of present results with experimental results of Isaakyan & Gasparyan. $(V-v)_{I,1}$, Isaakyan & Gasparyan, $R = 0{\cdot}080$; $(V-v)_{I,2}$, Isaakyan & Gasparyan, $R = 0{\cdot}160$; V_1-v_1, present results, $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$; V_2-v_2, present results, $0{\cdot}085 \leqslant R \leqslant 0{\cdot}150$; V_3-v_3, present results, $0{\cdot}160 \leqslant R \leqslant 0{\cdot}216$.

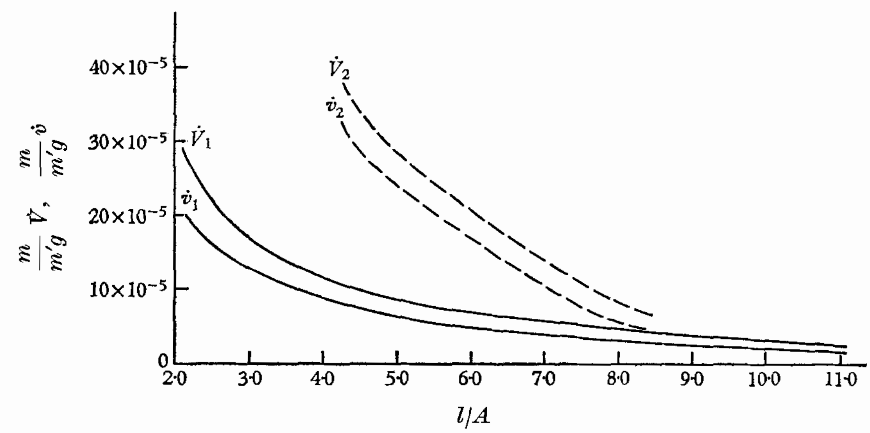

FIGURE 4. Variation of the acceleration with distance between spheres of a pair. Present results. $\dot{V}_1$, upper sphere; $\dot{v}_1$, lower sphere; $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$. $\dot{V}_2$, upper sphere; $\dot{v}_2$, lower sphere; $0{\cdot}085 \leqslant R \leqslant 0{\cdot}140$.

Since R is quite small for all three groups, the expectation would be that Stokes's approximation, and therefore the Stimson–Jeffery solution for the motion of a pair of spheres, would be valid. As we have already seen, the experiments give different drag forces for the two spheres, contrary to the S–J theory. In figure 2 the 'xxx' line shows λ as evaluated by the S–J theory. It is practically coincident with λ_1, the drag coefficient for the lower sphere for group (i), for

$3{\cdot}6 \leqslant l/A \leqslant 7{\cdot}6$. For smaller l/A, λ_1 is less than Λ_{SJ} and for larger separations $\lambda_1 > \Lambda_{SJ}$. For all values of l/A $\Lambda_1 < \Lambda_{SJ}$, but the difference decreases with increase in l/A. For larger R, groups (*ii*) and (*iii*), the values of λ, Λ deviate more from Λ_{SJ} than λ_1 and Λ_1.

Oseen's approximate solutions for λ, Λ are also shown in figure 2, for three values of R. They depart from the Stimson–Jeffery solution qualitatively in the same way as our experimental results, namely lower at small l/A and higher at large l/A. Both in the experimental results and in the Oseen theory λ increases with R and the rate of variation of λ with l/A is almost the same for $l/A > 7{\cdot}0$, although the Oseen values are smaller than the corresponding experimental values. For small l/A the difference between the Oseen solutions and the experiment are larger, but since the Oseen solution neglects terms of the order A^2/l^2,

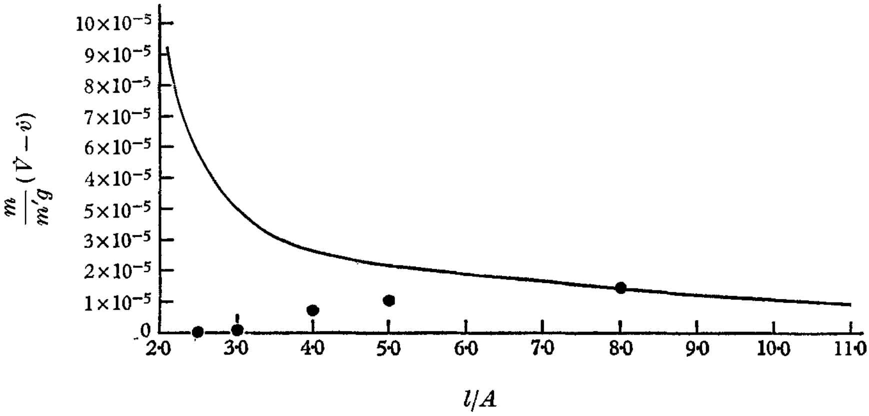

FIGURE 5. Difference between non-dimensional drag force (in units of $m'g$) on forward (lower) and rear (upper) sphere as function of distance between them. Present results (curve) compared with experimental results of Telford & Cottis. ——, present results, $0{\cdot}060 \leqslant R \leqslant 0{\cdot}085$; ●, Telford & Cottis, $R = 0{\cdot}088$.

large deviations are to be expected. In Oseen's solution Λ does not vary with Reynolds number, but the experiments showed that Λ increases with R, though not as much as λ.

The experiments by Hall (1956), Pfeffer (1958), Bart (1959), Eveson *et al.* (1959) and Happel & Pfeffer (1960), the results of which are shown in figure 1, were designed on the assumption that R would be so small that the two spheres would fall with equal constant speed, and thus only the average speed of one of them and their separation was measured. Thus they could not by the nature of their experiment find different speeds for the two spheres. Isaakyan & Gasparyan, however, found that the upper sphere fell faster than the lower, and our experiments corroborate this. Also, the first group of experimenters assumed that there would be no difference for different Reynolds numbers in the range they worked with and considered that the scatter of their data was due entirely to experimental uncertainty. However, when one looks at their data in the light of the Reynolds number effect found in our experiments one sees that some of the scatter is just the reflexion of the differences in R. In fact the various points agree remarkably well with our results, generally falling not far from our curve for the corresponding R.

Isaakyan & Gasparyan found the velocity of the lower sphere to depend on R, but that of the upper sphere to be independent of R, in accord with Oseen's theoretical expression. Their values are somewhat higher, particularly for small l/A, than ours. They carried out the experiment by taking motion pictures of the falling spheres, a method which is subject to fairly large errors in determining the successive positions of the spheres. The uncertainties of their method, together with the closeness of the measurements by the other investigators to ours, lead us to believe that the values we have obtained are more nearly correct.

5. Conclusions

The striking result of our experiments is that even for the smallest values of R ($\sim 0{\cdot}06$) the drag coefficient for the upper sphere is significantly smaller than that for the lower sphere. Thus the range of validity of the Stokes approximation is much more limited than commonly considered. This conclusion was reached previously with respect to the drag on a single sphere (Maxworthy 1965; Pruppacher & Steinberger 1968).

Another new result is the fact that the drag coefficient of the upper sphere varies with Reynolds number. Oseen's theory does not give this effect for line-of-centres motion, and previous experiments did not discover this deviation from Oseen, even for larger values of R. However, from the standpoint of physical intuition one would expect that the deviation from Stokes drag would be greater for the upper as well as the lower sphere for larger Reynolds number. This suggests that the failure of the Oseen theory to predict it is a deficiency of the model assumed or else of the method of approximation of the solution.

On the other hand, the fact that for the smallest R group λ is very close to Λ_{SJ} over a considerable range of distances indicates that the Stimson–Jeffery solution comes fairly close to representing the real situation for the lower sphere for this value of R. The equality of drag on the two spheres predicted by the Stokes approximation appears to require much smaller values of R, however.

The general behaviour of λ is quite well represented by the Oseen expression, even though the experimental values are somewhat larger. In particular, the variation with l/A is well represented for large l/A, and the variation with R is in the right sense. The variation of Λ with l/A is similarly represented, but of course its variation with R is not predicted in the Oseen solution.

The cross-over of sense of the variation of Λ and λ or V and v with R as l/A increases can be explained in terms of the difference in effect of the inertial terms in the fluid equation for large and small separations. For large l/A the effect of non-zero R is in the direction of increasing λ and Λ above unity, and the amount of increase is larger for larger R. As the distance between the spheres gets smaller, the effect of their interaction is to reduce λ and Λ, and the amount of reduction is greater for larger R. When they are sufficiently close together the reduction effect dominates and the cross-over occurs, with smaller λ, Λ for larger R.

While the Reynolds number of the smallest group we were able to study was somewhat higher than that of the most frequent cloud drops, the fact that for this group the upper sphere was constantly overtaking the lower one suggests that even for the smaller cloud drops equal drops may tend to coalesce. It will require further experimentation to find the limiting size for non-zero collision efficiencies of equal drops.

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